PHYS-2201 Electricity and Magnetism
Midterm 2 Study Problems
Note 1: Midterm 2 will be 15% of overall grade
Note 2: Midterm 2 will be Wed. Dec. 16, 2015, 9am (1.5 hours)

For the midterm you may use a calculator. No cell phones, computers, tablets, or other memory devices allowed. The midterm will have four problems with difficulties ranging from easy to hard. Please show all of your work in the booklets provided to get full marks.

1 Equation Sheet

Useful constants
$k_e = \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9$ N m^2/C^2, $\mu_0 = 4\pi \times 10^{-7}$ T m / A, $c = 2.998 \times 10^8$ m/s
$e = 1.602 \times 10^{-19}$ C, $m_e = 9.11 \times 10^{-31}$ kg, $m_p = 1.67 \times 10^{-27}$ kg,
$m_{\text{deuteron}} = 3.34 \times 10^{-27}$ kg
$N_A = 6.02 \times 10^{23}$, $g = 9.8$ m/s^2, $\rho_{\text{Cu}} = 1.7 \times 10^{-8}$ Ωm, $\rho_{\text{Al}} = 2.8 \times 10^{-8}$ Ωm
Current and resistance
$v_d = \frac{I}{nqA}$, $n = N_A \frac{\rho}{M}$
$J = \frac{I}{A} = \frac{\sigma}{E}$, $R = \frac{V}{I} = \frac{\rho l}{A} = \frac{l}{\sigma A}$, $\sigma = \frac{nq^2\tau}{m_e}$, $V = IR$, $P = I\Delta V = I^2 R = \frac{\Delta V^2}{R}$
Capacitance and capacitors
$C = \frac{Q}{V}$, $Q = CV$, $U_C = \frac{Q^2}{2C} = \frac{1}{2}CV^2$, $C = \kappa C_0$
parallel plates: $C = \frac{\epsilon_0 A}{d}$, concentric cylinders: $C = \frac{1}{2k_e \ln b/a}$, spherical: $C = \frac{ab}{k_e(b-a)}$
Electric Fields, potentials and forces
$dq = \rho dV = \sigma dA = \lambda dl$, $\vec{F}_e = q\vec{E}$, $\Sigma\vec{F} = m\vec{a}$
$\Delta V = \frac{U}{q}$, for constant E: $\Delta V = -Ed$
$W = U = -q_0 \int_a^b \vec{E} \cdot \vec{ds}$, $\Delta V = -\int_{\text{path}} \vec{E} \cdot \vec{ds}$, $U = \frac{\epsilon_0}{2} \int_{\text{volume}} E^2 dV$
$\vec{E}(x,y,z) = -\nabla V(x,y,z) = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$
electrostatic pressure: $p = \frac{\vec{F}}{A} = \frac{\sigma}{2}(\vec{E}_1 + \vec{E}_2)$
point charges: $\vec{F}_{12} = k_e \frac{q_1 q_2}{r^2}\hat{r}$, $\vec{E} = k_e \frac{q}{r^2}\hat{r}$, $\vec{E} = k_e \sum_i \frac{q_i}{r_i^2}\hat{r}_i$, $V = \frac{k_e q}{r}$ ($V = 0$ at ∞)
continuous charges: $\vec{E} = \int d\vec{E} = \int \frac{k_e dq}{r^2}\hat{r} = k_e \int \frac{dq}{|\vec{r}-\vec{r_i}|^3}(\vec{r} - \vec{r_i})$
Gauss' law: $\Phi_E = \oint_S \vec{E} \cdot d\vec{A} = \frac{q_{\text{encl.}}}{\epsilon_0}$
ring axis (x): $E = k_e \frac{Qx}{(x^2+a^2)^{3/2}}$, disk axis ($x$): $E = 2\pi k_e \sigma \left[1 - \frac{x}{\sqrt{x^2+R^2}}\right]$
dipole ($\vec{p} = q\vec{a}$): $\vec{E} = \frac{k_e}{r^3}(2p\cos\theta\hat{r} + p\sin\theta\hat{\theta})$, $\vec{\tau} = \vec{p} \times \vec{E}$, $\sum\vec{\tau} = I\frac{d^2\vec{\theta}}{dt^2}$, $U = -\vec{p} \cdot \vec{E}$
Maxwell's equations in vacuum
Gauss: $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$
Useful equations from classical mechanics
Simple harmonic motion: $\frac{d^2x}{dt^2} = -\omega^2 x$, $f = \frac{1}{T} = \frac{\omega}{2\pi}$, $\omega = \frac{v}{R} = 2\pi f$
centripital acceleration: $\vec{a} = \frac{v^2}{r}\hat{r}$

$x_f = x_i + v_i t + \frac{1}{2}at^2$, v_f=$v_i + at$, $v_f^2 = v_i^2 + 2a(x_f - x_i)$
Useful theorems and integrals
Gauss theorem: $\int_{\text{volume}} \vec{\nabla} \cdot \vec{F} dV = \int_{\text{surface}} \vec{F} \cdot d\vec{A}$
Integrals: $\int x^n dx = \frac{1}{n+1} x^{n+1} + c$, for $n \neq -1$, and $\int x^{-1} dx = \ln x + c$
$\int \cos x dx = \sin x + c$, $\int \sin x dx = -\cos x + c$, $\int \mathrm{e}^{ax} dx = \frac{\mathrm{e}^{ax}}{a} + c$
Binomial expansion: $(x + \Delta x)^n = x^n + n x^{n-1} \Delta x + \frac{n!}{2!(n-2)!} x^{n-2} \Delta x^2 + ...$
$(1 + x)^\alpha \approx 1 + \alpha x$, $|x| << 1$
Cylindrical coordinates (r, z, θ)
$x = r\cos\theta$, $y = r\sin\theta$, $z = z$
Volume element: $dV = r dr d\theta dz$
Path element: $\vec{ds} = dr\hat{r} + r d\theta \hat{\theta} + dz\hat{z}$
Gradient: $\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z}$
Divergence: $\nabla \cdot \vec{E} = \frac{1}{r}\frac{\partial (rE_r)}{\partial r} + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_z}{\partial z}$
Spherical coordinates (r, θ, ϕ)
θ is from z axis. $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$
Volume element: $dV = r^2 \sin\theta dr d\theta d\phi$
Path element: $\vec{ds} = dr\hat{r} + r d\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$
Gradient: $\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$
Divergence: $\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial (r^2 E_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(E_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial \phi}$

2 Sample Problems

1. An iron wire has a cross-sectional area A. Determine the drift speed of the conduction electrons when there is a current I flowing in the wire. Express your answer using the variables ρ_I for the mass density of iron, and M_I for the molar mass of iron (in g/mol).

2. Suppose a thin copper wire with radius a is coated with a layer of gold making the overall radius of the wire b. What is the resistance of a length L of this wire?

3. A Van de Graff generator produces a beam of deuterons (heavy hydrogen nuclei containing one proton and one neutron) to a kinetic energy T_d. If the beam current (in Amperes) is I_d, what is the average separation of the deuterons? Once you have your solution as an equation use $I_d = 18$ μA, and $T_d = 1.5$ MeV. Is the electrical force of repulsion among the deuterons significant in the beam stability?

4. A rod has charge per unit length $\lambda = Ax^2$, and extends from $x = b$ to $x = 5b$.

   (a) What units must A have?

   (b) What is the total charge on the rod?

5. A positively charged bead q can slide without friction around a vertical hoop of radius R. A fixed positive charge Q is at the bottom of the hoop. Define the counterclockwise angular displacement of q relative to Q to be θ. What is the electrical force on q as a function of θ?

6. A cylindrical shell of radius R, and length L, has a surface charge density of σ. Determine the electric field along the axis of the shell a distance $z > 0$ from one end of the cylinder.

7. A cylinder of radius a and length l has a charge per unit volume $\rho = C + Br^2$.
   (a) What are the units of B and C?
   (b) Find the charge dq in a cylindrical shell of radius r and thickness dr.
   (c) Find the total charge Q on the whole cylinder.

8. A sheet of Mylar with dielectric constant κ_m and breakdown voltage of E^m_{max}, is painted on one side with titanium dioxide with dielectric constant κ_t and breakdown voltage of E^t_{max}. The Mylar has a thickness d_m, and the titanium dioxide layer is d_t thick. This stack of dielectrics is placed between two flat conducting plates to make a capacitor.
   (a) What is the area of each conducting plate that is required to make a capacitor with capacitance, C?
   (b) What fraction of the potential, V, applied across the stack of dielectrics is across each layer of dielectric?

9. A long rod of charge per unit length $\lambda > 0$ is normal to the xy-plane and passes through the origin. in addition, a charge $Q > 0$ is located at $(0, a, 0)$. Find the position where a third charge q will feel zero force.

10. A rod of length a whose ends are at $(0, 0)$ and $(a, 0)$ has a charge density $\lambda = (Q_0/a^2)x$. (a) Find the total charge Q on the rod. (b) Find the force on a charge q at $(-b, 0)$. (c) What is the force on the charge in the limit $b >> a$?

11. A charge Q is uniformly distributed over an arc of radius a and angle α that extends from the x-axis counterclockwise. Find the force on a charge q located at the center of the semicircle.

12. Consider a uniformly charged rod of two pieces from (a, a) to $(3a, a)$ and from $(-3a, a)$ to $(-a, a)$. Each piece has charge $Q/2$. Find the electric field at the origin (using superposition).

13. Sketch the electric field lines for two line charges λ and -4λ that are normal to the page, and separated by a distance a. Draw four lines per λ. Find the position where the electric field is zero.

14. A dipole $\vec{p}$ points along the x-axis. It is in a field $\vec{E} = Ax\hat{\imath} + Ay\hat{\jmath} - 2Az\hat{k}$. Find an expression for the force on the diple in terms of $|\vec{p}|$ and A.

15. Consider two horizontal plates with equal and opposite charge densities $\pm\sigma$ located at $z = +d/2$ and $z = -d/2$. An electron enters this region at its left edge at with $y = z = 0$ alon the midline with a velocity $v\hat{\imath}$. The plates have a length L, and separation d.

    (a) Find the electric field at which the electron just misses hitting either plate.

    (b) Find the charge density corresponding to this field.

    (c) Find the velocity of the electron as it leaves the two plates.

16. What volume charge density $\rho(r)$ is needed to make the magnitude of the electric field constant inside an infinitely long cylinder of radius R? The cylinder has total charge per unit length of Q/L. What is the electric field inside the cylinder in terms of this charge density?

17. Consider two thin concentric conducting shells of radius $r_1 < r_2$ with positive charges Q_1 and Q_2.

    (a) Find the charge densities on their inner and outer surfaces.

    (b) If a charge Q_3 is brought to just outside of r_2, how does this affect the charge density, qualitatively (no calculation needed).

18. Find the electrical pressure on a cylindrical shell of radius r and charge per unit length λ. Determine the work done against the electrical force to compress it from radius b to radius a.

19. Three thin conducting plates are normal to the x-axis. The one on the left has surface charge density 2σ, the middle one has a surface charge density -4σ, and the right one has surface charge density -3σ.

    (a) Find the electric field in the different regions.

    (b) What are the charge densities on each surface of the conductors?

20. Let $\vec{E} = (ax + b)\hat{\imath}$.

    (a) Find the circulation $\oint \vec{E} \cdot \vec{ds}$ for a square path going clockwise from $(0, 0)$ to $(0, d)$, then to (d, d), then to $(d, 0)$, and finally back to $(0, 0)$.

    (b) Is this what is expected for electrostatics?

    (c) If so, what is $V(x, y) - V(0)$?

21. Find the potential a distance z along the axis of an annulus of inner radius a and outer radius b with uniform charge density σ.

22. From the dipole potential $V = k_e \vec{p} \cdot \vec{r}/r^3$, show that the electric field due to a dipole located at the origin is given by:

$$\vec{E} = \frac{k_e}{r^3}[3(\vec{p} \cdot \vec{r})\vec{r} - \vec{p}] \tag{1}$$

23. Let $V = 2x - 5x^2 - 5x^3$.

    (a) Find the electric field $\vec{E}$.

    (b) Find the electric flux leaving a unit cube in the first octant. Does this correspond to the presence of electric charge?

24. Consider two conducting co-axial rings of radii a with equal and opposite charges $\pm q$. their ceners are at $\pm h/2$ along the z-axis. h is to be adjusted to make a very uniform field in the vicinity of the midpoint between the grings. Consider V along the z-axis.

    (a) Show that at $x = 0$ all the even derivatives of $V(0, 0, z)$ are zero.

    (b) Find the value of h/a that makes the third derivative go to zero.

25. Let $V(r) = V_0 \mathrm{e}^{-\mathrm{r}/\mathrm{a}}$ in a spherical coordinates.

    (a) Find E_r.

    (b) Find Q_{enc} for a Gaussian surface that is a concentric sphere of radius b.

    (c) Find the charge density $\rho(r)$.

26. (a) Show that the energy needed to assemble a spherical shell of radius R and total charge Q by bringing the charge from infinity bit by bit is $U = k_e Q^2/2R$. Hint: integrate over dQ' from 0 to Q using $dU = V dQ$, where $V = kQ'/R$.

    (b) Show that the pressure needed to keep the sphere from expanding is $p = k_e Q^2/2R^2$. Hint: relate a change in electrical energy to the radial component of the electric force.

27. Consider a spherical shell of inner radius a and outer radius b, its upper half filled with air, and its lower half filled with dielectric, κ. With $\kappa_{air} \approx 1$, what is the capacitance? Hint: Draw the field lines before the dielectric is added. Will adding a dielectric change the direction of the field lines? Will polarization charge appear along the surface that separates the air and the dielectric?

28. Capacitors C_A and C_B are connected in series with a power supply providing a potential V.

    (a) Find the potential across each of the capacitors, the charge on each capacitor, and the energy stored by each capacitor.

(b) The power supply is removed and the plates of like sign (+ to +, and - to -) are connected. Find the potential, charge and energy stored on each capacitor.

29. Consider the earth to be a spherical capacitor with $a = 6.37 \times 10^6$ m and $b = a + d$, where $d = 5 \times 10^4$ m (the distance to the ionosphere). If the electric field is 100 V/m, find the energy stored in the earth's electric field.